数理解析研究所講究録
1024巻 1998年 55-60

Actions of Compact Quantum Groups on Operator Algebras

Masaki Izumi
Department of Mathematical Sciences
University of Tokyo
Komaba, Meguro-ku, Tokyo 153, Japan

1 Introduction

The notion of compact quantum groups, matrix pseudogroups in original terminology, was first introduced by S. L. Woronowicz on the basis of C^*-algebra theory [22], and it is the dual notion of Drinfel'd and Jimbo's quantum universal enveloping algebras [4][7]. Since it may provide a new kind of symmetry because it generalizes the notion of ordinary groups, actions of quantum groups on operator algebras have drawn several authors' attention. In this note we report recent results of actions, especially product type actions, of compact quantum groups on operator algebras.

Among other results, we focus on the relationship between quantum probability theory and product type actions, which is an unexpected byproduct of the subject. Usually, product type actions of ordinary compact groups are typical examples of so called minimal actions, which mean the triviality of the relative commutants of the fixed point algebras. However, the natural product action of $SU_q(2)$ on the Powers factor does not have this property; the Podles quantum sphere arises as the relative commutant. The mathematical structure behind this phenomenon is parallel to the boundary theory of random walks on discrete groups, and here the role of the discrete group is replaced with the dual Hopf algebra of $SU_q(2)$.

2 Product Type Actions

Since we do not need the general definition of compact quantum groups, we just give that of $SU_q(2)$ introduced by Woronowicz [21]. For general theory, see [22]. Our choice of generators is taken from [13].

Definition 2.1 *Let q be a non-zero real number satisfying $|q| \leq 1$. $C(SU_q(2))$ is the universal C^*-algebra generated by four elements x, u, v, and y satisfying the following relations:*

$$ux = qxu, \quad vx = qxv, \quad yu = quy, \quad yv = qvy,$$

$$uv = vu, \quad xy - q^{-1}uv = yx - quv = 1,$$

$$x^* = y, \quad u^* = -q^{-1}v.$$

Let (w_{ij}) be the matrix with entries in $C(SU_q(2))$ defined by

$$\begin{pmatrix} x & u \\ v & y \end{pmatrix}.$$

Thanks to the universality of $C(SU_q(2))$, there exists a $*$-homomorphism, called the coproduct,

$$\Delta : C(SU_q(2)) \longrightarrow C(SU_q(2)) \otimes_{min} C(SU_q(2))$$

determined by the following relations:

$$\Delta(w_{ij}) = \sum_k w_{ik} \otimes w_{kj}.$$

Therefore, $C(SU_q(2))$ is a matrix pseudogroup in the sense of Woronowicz [22]. There exists a unique invariant state on $C(SU_q(2))$ called the Haar measure. We denote by $L^\infty(SU_q(2))$ the weak closure of $C(SU_q(2))$ in the GNS representation of the Haar measure.

Although the notion of actions of quantum groups is fairly general, we introduce it just for $SU_q(2)$, which is enough for our purpose.

Definition 2.2 *A (right) action* Γ *of* $SU_q(2)$ *on a* C^**-algebra* A *is a* $*$*-homomorphism* $\Gamma : A \longrightarrow A \otimes_{min} C(SU_q(2))$ *satisfying*

$$(\Gamma \otimes id) \cdot \Gamma = (id \otimes \Delta) \cdot \Gamma.$$

A (right) action Γ *of* $SU_q(2)$ *on a von Neumann algebra* M *is a normal* $*$*-homomorphism* $\Gamma : M \longrightarrow M \otimes L^\infty(SU_q(2))$ *satisfying*

$$(\Gamma \otimes id) \cdot \Gamma = (id \otimes \Delta) \cdot \Gamma.$$

In a similar way, one can introduce left actions just changing the order of tensor product in an appropriate way.

Let A, M, Γ be as above. We say that $x \in A$ (resp. $x \in M$) is invariant under Γ if $\Gamma(x) = x \otimes 1$, and denote by A^Γ (resp. M^Γ) the set of invariant elements. A^Γ is called the fixed point subalgebra of A under Γ.

Let A be the UHF algebra of type 2^∞, which is the infinite tensor product of 2 by 2 matrix algebra M_2:

$$A = \otimes_{i=1}^\infty M_2.$$

The infinite tensor product action of $SU_q(2)$ on A was introduced by Y. Konishi, M. Nagisa and Y. Watatani [12] as follows: Let $\{e_{ij}^{(k)}\}_{ij}$ be a system of matrix unit of the kth tensor component. We define unitary operators $V^{(k)}$ and $W^{(k)}$ in $A \otimes C(SU_q(2))$ by

$$V^{(k)} = \sum_{ij} e_{ij}^{(k)} \otimes w_{ij}, \quad W^{(k)} = V^{(1)}V^{(2)} \cdots V^{(k)}.$$

Then we can define an action Γ of $SU_q(2)$ by the following limit:

$$\Gamma(x) = \lim_{n\to\infty} Ad(W^{(n)})(x \otimes 1), \quad x \in A.$$

Thanks to the q-version of the Weyl duality theorem, we can show that the fixed point algebra A^Γ is generated by Jones projections (R-matrices) with the index parameter $(q+q^{-1})^2$. (See [8] for Jones projections).

There are several interesting observations about quantum group actions on UHF algebras made by M. Fannes, B. Nachtergaele, and R. F. Werner [3]. Among others, one of their results shows that the above one-side infinite tensor product action is the best possible generalization of infinite tensor product actions of compact groups. Namely, they prove that there is no non-trivial translation invariant action of a proper quantum group on two-side infinite tensor product.

3 Minimal Actions

In what follows, we assume $q \neq 1$. In [11], Konishi shows that one of the Powers states, which is the infinite product state of so called the normalized q-trace in 2-dimensional representation of $SU_q(2)$, is an invariant state of the action introduced in the previous section. As in the case of usual group actions, one can extend the action of $SU_q(2)$ to the weak closure of the UHF algebra A in the GNS representation of the invariant Powers state, which is denoted by R_{q^2}. For simplicity, we use the same symbol Γ for the extended action. Nakagami generalizes this construction to $SU_q(N)$ case, and investigates the structure of the corresponding crossed products [14].

An action Γ of a quantum group on a factor M is called minimal if the relative commutant $M \cap M^{\Gamma\prime}$ is trivial. Typical examples of minimal actions of compact groups come from infinite tensor product actions with infinite product invariant states. However, unlike the classical case our action on R_{q^2} is not minimal. Indeed, if it were minimal, it is not so difficult to show that the subfactor $R_{q^2}^\Gamma \supset \sigma(R_{q^2}^\Gamma)$ would be irreducible, i.e. $R_{q^2}^\Gamma \cap \sigma(R_{q^2})^{\Gamma\prime} = \mathbf{C}$, where σ is the shift endomorphism. However, this inclusion is nothing but the Jones inclusion with index larger than 4, which is well-known to be not irreducible [8]. The same argument works for $SU_q(N)$ case [18].

In view of the above example, it is tempting to conjecture that there is no faithful minimal action of non-Kac compact quantum groups on AFD factors because for AFD factors, product type actions are somehow believed to be universal objects. However, if AFD condition is removed, there is a counter example due to Y. Ueda based on the free product method. In [19], he constructs, among other things, a minimal action of $SU_q(N)$ on a full type III_{q^2} factor. Note that since the Haar measure of $SU_q(2)$ is not a trace state, it is not so difficult to show that there is no faithful minimal action of $SU_q(N)$ on type II factors.

4 Relative Commutants

As we saw in the previous section, the relative commutant $\mathcal{B} = R_{q^2} \cap R_{q^2}^{\Gamma\prime}$ is not trivial. It is easy to show that the restriction of Γ to $\mathcal{B}$ is again an action of $SU_q(2)$, which is ergodic in the sense that the fixed point algebra is trivial. We would like to show how to describe $\mathcal{B}$ both as an algebra and as an $SU_q(2)$-space. It turns out that a non-commutative version of the theory of Poisson boundaries of random walks plays a crucial role in the description. For the classical theory of Poisson boundaries of random walks, see [9][10][20]. Note that it has already played an essential role in the index theory of operator algebras [1][2][5][16][17].

Let $\ell^\infty(\widehat{SU_q(2)})$ be the dual Hopf algebra of $L^\infty(SU_q(2))$, and $\hat{\Delta}$ the dual coproduct. As a von Neumann algebra, $\ell^\infty(\widehat{SU_q(2)})$ is isomorphic to the group von Neumann algebra of $SU(2)$. We introduce a non-commutative Markov operator P on $\ell^\infty(\widehat{SU_q(2)})$, which is a completely positive map, by $P = (id \otimes \tau_q) \cdot \hat{\Delta}$, where τ_q is the normalized q-trace as before. We denote by $H^\infty(\widehat{SU_q(2)}, P)$ the set of fixed elements under P, which we call harmonic elements with respective to P. Note that $H^\infty(\widehat{SU_q(2)}, P)$ is *not* an algebra but an operator system.

Using the non-commutative martingale convergence theorem, we can show the following:

Theorem 4.1 **([6])** *There is a surjective isometry $\theta : H^\infty(\widehat{SU_q(2)}, P) \longrightarrow \mathcal{B}$ which intertwines the natural actions of $SU_q(2)$. Moreover, one can recover the product structure of $\mathcal{B}$ from $H^\infty(\widehat{SU_q(2)}, P)$ and P by the following formula:*

$$\theta^{-1}(\theta(x)\theta(y)) = s - \lim_{n\to\infty} P^n(xy), \quad x, y \in H^\infty(\widehat{SU_q(2)}, P).$$

In view of the classical case [9], this result indicates that $\mathcal{B}$ should be interpreted as the "function space" on the "Poisson boundary" of the "quantum random walks" induced by P. Moreover, the next result shows that the "Poisson boundary" should be $\mathbf{T}\backslash SU_q(2)$.

In [15] P. Podles introduced a family of quantum spheres, which are C^*-algebras with ergodic $SU_q(2)$ actions satisfying a certain spectral condition under the actions. The homogeneous space $C(\mathbf{T}\backslash SU_q(2)) \subset C(SU_q(2))$ is the most natural one among them. Let $L^\infty(\mathbf{T}\backslash SU_q(2))$ be the weak closure in the GNS representation with respective to the unique $SU_q(2)$-invariant state.

By using the representation theory of $SU_q(2)$ and random walks on $\mathbf{N}$, we can determine the structure of $\mathcal{B}$ through $H^\infty(\widehat{SU_q(2)}, P)$ and P.

Theorem 4.2 **([6])** *There is an isomorphism between $\mathcal{B}$ and $L^\infty(\mathbf{T}\backslash SU_q(2))$ that intertwines the natural $SU_q(2)$-actions.*

There is a natural left $\widehat{SU_q(2)}$ action on $L^\infty(\mathbf{T}\backslash SU_q(2))$, which is a "purely quantum" phenomenon because it is trivial when $q = 1$. The natural map between $L^\infty(\mathbf{T}\backslash SU_q(2))$ and $H^\infty(\widehat{SU_q(2)}, P)$, obtained by composing the two maps in the above theorems, can be given by an explicit formula with the Haar measure and the multiplicative unitary. Using this formula, one can show that the map intertwines the natural left $\widehat{SU_q(2)}$ actions as well as the right $SU_q(2)$ actions. The formula can be interpreted as generalization of the Poisson integral formula in [9].

One might wonder why all these phenomena occur only when $q \neq 1$. Although there is no philosophical explanation so far, it is worth pointing out the fact that $\mathbf{T}\backslash SU_q(2)$ is the very deformed part of $SU_q(2)$ while the maximal torus $\mathbf{T}$ remains undeformed. The difference between $q = 1$ case and $q \neq 1$ case appearing in the proofs is as follows. It often occurs that some quantities, which are functions of q and the spin l of irreducible representations, have completely different asymptotic behavior as l goes to infinity; in one case it has polynomial growth while in the other case it has exponential growth.

There are two directions of generalizing the results stated in this section. One is to replace the fundamental representation of $SU_q(2)$ with other representations. The other is to replace $SU_q(2)$ with other quantum groups, for the first step, say $SU_q(N)$. Probably it is not so difficult to do the former, and the result should be the same. On the other hand, since our analysis highly depends on the representation theory of $SU_q(2)$, our method works only for $SU_q(2)$ so far. The Poisson integral formula mentioned above might play some role in this case because it is given by a general formula which works for very compact quantum group.

References

[1] Bisch, D.: Entropy of groups and subfactors. J. Funct. Anal. **103**, 190-208 (1992)

[2] Bisch, D., Haagerup, U.: Composition of subfactors: new examples of infinite depth subfactors. Ann. Sci. École Norm. Sup. Sér. 4, **29**, 329-383 (1996)

[3] Fannes, M., Nachtergaele, B., Werner, R. F.: Quantum spin chains with quantum group symmetry. Commun. Math. Phys. **174**, 477-507 (1996)

[4] Drinfel'd V.G.: Quantum groups. In: Vol. I of the Proceedings of the Int. Nat. Congr. Math. Berkeley 1986, New York: Academic Press 1987, pp. 798-820

[5] Hiai, F., Izumi, M.: Amenability and strong amenability for fusion algebras with applications to subfactor theory. preprint, 1996

[6] Izumi, M. in preparation

[7] Jimbo, M.: A q-difference analogue of $U(g)$ and the Yang-Baxter equation. Lett. Math. Phys. **10**, 63-69 (1985)

[8] Jones, V. F. R.: Index for subfactors. Invent. Math. **72**, 1-25 (1983)

[9] Kaimanovich, V. A.: Measure-theoretic boundaries of Markov chains, 0-2 laws and entropy. In: Harmonic Analysis and Discrete Potential Theory, M. A. Picardello (ed.), New York: Plenum Press, 1992, pp. 145-180.

[10] Kaimanovich, V. A., Vershik, A. K.: Random walks on discrete groups: boundary and entropy. Ann. Probab. **11**, 457-490 (1983)

[11] Konishi, Y.: A note on actions of compact matrix quantum groups on von Neumann algebras. Nihonkai Math. J. **3**, 23-29 (1992)

[12] Konishi, Y., Nagisa, M., Watatani, Y.: Some remarks on actions of compact matrix quantum groups on C^*-algebras. Pacific J. Math. **153**, 119-127 (1992)

[13] Masuda, T., Mimachi, K., Nakagami, Y., Noumi, M.: Representation of the quantum group $SU_q(2)$ and the Little q-Jacobi Polynomials. J. Funct. Anal. **99**, 357-386 (1991)

[14] Nakagami, Y.: Tankesaki's duality for the crossed product by quantum groups. In: Quantum and non-commutative analysis, H. Araki et al (ed.), Kluwer Academic Publishers, 1993, pp. 263-281

[15] Podles, P.:Quantum Spheres. Lett. Math. Phys. **14**, 193-202 (1987)

[16] Popa, S.: Sousfacteurs, actions des groupes et cohomologie, C. R. Acad. Sci. Paris, Sér I. **309**, 771-776 (1989)

[17] Popa, S.: Classification of amenable subfactors of type II, Acta Math. **172**, 163-255 (1994)

[18] Sawin, S.: Relative commutants of Hecke algebra subfactors. Amer. J. Math. **116**, 591-604 (1994)

[19] Ueda Y.: A minimal action of the compact quantum groups $SU_q(2)$ on a full factor. preprint, 1997

[20] Woess, W.: Random walks on infinite graphs and groups – a survey on selected topics. Bull. London Math. Soc. **26**, 1-60 (1994)

[21] Woronowicz, S.L.: Twisted $SU(2)$ group. An example of a non-commutative differential calculus. Publ. RIMS, Kyoto Univ. **23**, 117-181 (1987)

[22] Woronowicz, S.L.: Compact matrix pseudogroups. Commun. Math. Phys. **111**, 613-665 (1987)